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NORMAL FAMILIES CONCERNING SHARED VALUES

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ABSTRACT

Let $\mathfrak{F}$ be a family of holomorphic functions in the unit disk D. Suppose that there exists a nonzero and finite value a such that for each function $f \in \mathfrak{F}$, f, f' and f'' share the value a IM in D. Then the family $\mathfrak{F}$ is normal in D. An example shows that a cannot be zero.

1. Introduction and main result

According to Bloch's principle, many normality criteria can be proved by starting from Picard type theorems (see [9]). Another approach to normality criteria is to use conditions known from theorems on sharing values. A first attempt at this was made by Schwick [10].

Let f and g be two meromorphic functions in the domain U, $a \in \mathbb{C}$. If $f - a$ and $g - a$ have the same zeros in U, then we say that f and g share the value a IM (ignoring multiplicity) in U (cf. [11]).

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THEOREM A ([10]): *Let $\mathfrak{F}$ be a family of meromorphic functions in the unit disk D and a_1, a_2, a_3 be distinct complex numbers. If f and f' share a_1, a_2, a_3 IM in D for every $f \in \mathfrak{F}$, then $\mathfrak{F}$ is normal in D.*

In this paper, we shall prove

THEOREM: *Let $\mathfrak{F}$ be a family of holomorphic functions in the unit disk D. Suppose that there exists a nonzero and finite value a such that for each function $f \in \mathfrak{F}$, f, f' and f'' share the value a IM in D. Then the family $\mathfrak{F}$ is normal in D.*

Remark 1: The following example shows that the value a cannot be zero.

Example: Let $\mathfrak{F} = \{f_n(z) = e^{nz} : n = 1, 2, ...\}$. Then the spherical derivative $f_n^{\#}(0) = n/2 \to \infty$. Thus $\mathfrak{F}$ is not normal in the unit disk D by Marty's criterion. However, it is clear that f_n, f_n' and f_n'' share 0, since none of these functions vanishes.

2. Preliminaries

We shall use standard notations in Nevanlinna theory (cf. [2]). Define

$$\Psi(z) := \psi(f(z)) = \frac{f'(z) + f''(z)}{f(z) - a} - \frac{2f''(z)}{f'(z) - a}. \tag{1}$$

Then $\psi(f(z)) \not\equiv 0$ implies that $f \not\equiv f'$.

For convenience, we set

$$\begin{aligned} LD(r, f : c_1, c_2, c_3, c_4) = & c_1 m\left(r, \frac{f'}{f-a}\right) + c_2 m\left(r, \frac{f''}{f'}\right) \\ & + c_3 m\left(r, \frac{f''}{f'-a}\right) + c_4 m\left(r, \frac{f''}{f-a}\right). \end{aligned}$$

We denote by M a positive number depending on a only, which may have different values at different occurrences.

LEMMA 1: *Let f be holomorphic in the unit disk D and $a \in \mathbb{C}\backslash\{0\}$. Suppose that f, f' and f'' share a IM in D. Then $f(z_0) = a$ implies $\Psi(z_0) = 0$.*

Proof: By the assumptions we may suppose that, near z_0,

$$f(z) = a + a(z - z_0) + \frac{a}{2}(z - z_0)^2 + b(z - z_0)^3 + O((z - z_0)^4),$$

where b is a constant relating z_0. Then we have

$$\begin{aligned} f'(z) =& a + a(z-z_0) + 3b(z-z_0)^2 + O((z-z_0)^3), \\ f''(z) =& a + 6b(z-z_0) + O((z-z_0)^2). \end{aligned}$$

Hence

$$\begin{aligned} \frac{f'+f''}{f-a} =& \frac{2}{z-z_0} + \frac{6b}{a} + O(z-z_0), \\ \frac{f''}{f'-a} =& \frac{1}{z-z_0} + \frac{3b}{a} + O(z-z_0). \end{aligned}$$

Thus $\Psi(z_0) = 0$. The proof of the lemma is complete. ∎

Lemma 2: *Let f be holomorphic in the unit disk D. Suppose that f, f' and f'' share a nonzero and finite value a IM in D. If $f(0) \neq a$ and $f''(0) \neq 0$, then*

$$T(r,f) \leq 2\overline{N}\left(r, \frac{1}{f-a}\right) + LD(f:1,2,1,0) + \log\frac{|(f(0)-a)(f'(0)-a)|}{|f''(0)|} + M.$$

Proof: From the assumptions we see that $f'(0) \neq a$. By the first and the second fundamental theorems,

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f'-a}\right) \leq& m\left(r, \frac{f'}{f-a}\right) + m\left(r, \frac{1}{f'}\right) + m\left(r, \frac{1}{f'-a}\right) \\ \leq& m\left(r, \frac{1}{f''}\right) + LD(f:1,1,1,0) + M \\ \leq& T(r,f'') + LD(f:1,1,1,0) + \log\frac{1}{|f''(0)|} + M \\ \leq& T(r,f') + LD(f:1,2,1,0) + \log\frac{1}{|f''(0)|} + M. \end{aligned}$$

Thus

$$\begin{aligned} T(r,f) \leq& N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f'-a}\right) + LD(f:1,2,1,0) \\ &+ \log\frac{|(f(0)-a)(f'(0)-a)|}{|f''(0)|} + M. \end{aligned}$$

Since f, f' and f'' share the value a, we know that $f-a$ and $f'-a$ have only simple zeros and

$$N\left(r, \frac{1}{f-a}\right) = N\left(r, \frac{1}{f'-a}\right).$$

The conclusion follows. ∎

LEMMA 3: *Let f be holomorphic in the unit disk D. Suppose that f, f' and f'' share a nonzero and finite value a IM in D. If $f(0) \neq a$, $f''(0) \neq 0$ and $\Psi(0) \neq 0$, then*

$$T(r,f) \leq LD(f:3,2,3,2) + \log \frac{|(f(0)-a)(f'(0)-a)|}{|f''(0)\Psi(0)^2|} + M.$$

Proof: From Lemma 1 we see that $\Psi(z)$ is holomorphic in D, and

$$\begin{aligned} N\left(r,\frac{1}{f-a}\right) \leq & N\left(r,\frac{1}{\Psi}\right) \\ \leq & T(r,\Psi) + \log \frac{1}{|\Psi(0)|} \\ \leq & LD(f:1,0,1,1) + \log \frac{1}{|\Psi(0)|} + M. \end{aligned}$$

This and Lemma 2 yield the conclusion. ∎

Remark 2: If f is a nonconstant entire function and f, f' and f'' share a finite and nonzero value a IM in the plane, then the above lemma implies that $f = f'$. This conclusion was already obtained by Jank–Mues–Volkmann [4]. Our proof is very simple.

The following result is the well-known Zalcman's principle.

LEMMA 4 ([12]): *Let $\mathfrak{F}$ be a family of meromorphic functions on the unit disk Δ. Then $\mathfrak{F}$ is not normal at $z=0$ if and only if there exists a sequence $f_n \subset \mathfrak{F}$, a sequence $z_n \to 0$ and a positive sequence $\rho_n \to 0$ such that $g_n(\zeta) = f_n(z_n + \rho_n\zeta)$ converges locally and uniformly to a non-constant entire function $g(\zeta)$.*

LEMMA 5 (see Hiong [3]): *If $f(z)$ is meromorphic in a disk $|z| < R$ such that $f(0) \neq 0, \infty$, then, for $0 < r < \rho < R$,*

$$m\left(r,\frac{f^{(k)}}{f}\right) \leq C_k\Big\{1 + \log^+\log^+\frac{1}{|f(0)|} + \log^+\frac{1}{r} + \log^+\frac{1}{\rho - r} + \log^+\rho + \log^+ T(\rho,f)\Big\},$$

where C_k is a constant depending only on k.

LEMMA 6 (Bureau [1]): *Let b_1, b_2 and b_3 be positive numbers and $T(r)$ a non-negative, increasing and continuous function on an interval $[r_0, R)$, $R < \infty$. If*

$$T(r) \leq b_1 + b_2\log^+\frac{1}{\rho - r} + b_3\log^+ T(\rho)$$

for any $r_0 < r < \rho < R$, *then*

$$T(r) \leq B_1 + B_2 \log^+ \frac{1}{R-r}$$

for $r_0 \leq r < R$, *where* B_1 *and* B_2 *depend only on* b_i $(i = 1, 2, 3)$.

3. Proof of the theorem

Suppose on the contrary that the family $\mathfrak{F}$ is not normal in D. Without loss of generality, we may suppose that $\mathfrak{F}$ is not normal at 0. By Zalcman's principle, there exist a sequence f_n in $\mathfrak{F}$, a sequence $z_n \to 0$ and a positive sequence $\rho_n \to 0$ such that

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta) \tag{2}$$

tends to a nonconstant entire function $g(\zeta)$ uniformly on compact subsets of $\mathbb{C}$. Thus, for any positive integer k,

$$g_n^{(k)}(\zeta) = \rho_n^k f_n^{(k)}(z_n + \rho_n\zeta) \to g^{(k)}(\zeta). \tag{3}$$

If g is a polynomial, then there exists a point w_0 such that $g(w_0) = a$. By Hurwitz' theorem, there is a sequence $\zeta_n \to w_0$ such that

$$g_n(\zeta_n) = f_n(z_n + \rho_n\zeta_n) = a \quad \text{for } n = 1, 2, \ldots.$$

Thus

$$f_n'(z_n + \rho_n\zeta_n) = f_n''(z_n + \rho_n\zeta_n) = a \tag{4}$$

for $n = 1, 2, \ldots$, since f_n, f_n' and f_n'' share a. By (3), we have

$$g_n'(\zeta_n) = \rho_n f_n'(z_n + \rho_n\zeta_n) \to g'(w_0)$$

and

$$g_n''(\zeta_n) = \rho_n^2 f_n''(z_n + \rho_n\zeta_n) \to g''(w_0).$$

This and (4) imply that

$$g'(w_0) = g''(w_0) = 0.$$

Thus $g(\zeta)$ is not a polynomial of degree less than 3.

Now there are two cases to be discussed.

CASE 1: There is a subsequence $\{f_{n_j}\} \subset \{f_n\}$ such that $\psi(f_{n_j}) \equiv 0$. Then by (1),

$$\frac{\rho_{n_j}^2 f'_{n_j}(z_{n_j}+\rho_{n_j}\zeta)+\rho_{n_j}^2 f''_{n_j}(z_{n_j}+\rho_{n_j}\zeta)}{f_{n_j}(z_{n_j}+\rho_{n_j}\zeta)-a} = \frac{2\rho_{n_j}^3 f''_{n_j}(z_{n_j}+\rho_{n_j}\zeta)}{\rho_{n_j} f'_{n_j}(z_{n_j}+\rho_{n_j}\zeta)-\rho_{n_j}a}.$$

Thus by (2),

$$\frac{\rho_{n_j} g'_{n_j}(\zeta)+g''_{n_j}(\zeta)}{g_{n_j}(\zeta)-a} = \frac{2\rho_{n_j} g''_{n_j}(\zeta)}{g'_{n_j}(\zeta)-a\rho_{n_j}}.$$

Letting $j\to\infty$, by (3), we obtain $g''(\zeta)\equiv 0$, which is a contradiction.

CASE 2: There are only finitely many f_n such that $\psi(f_n)\equiv 0$. We may suppose that $\psi(f_n)\not\equiv 0$ for all n. Take a point ζ_0 such that

$$g(\zeta_0)\neq a,0;\quad g'(\zeta_0)\neq 0;\quad g''(\zeta_0)\neq 0. \tag{5}$$

The same reason as above gives

$$\begin{aligned}\rho_n^2\psi(f_n(z_n+\rho_n\zeta_0)) =& \frac{\rho_{n_j} g'_{n_j}(\zeta_0)+g''_{n_j}(\zeta_0)}{g_{n_j}(\zeta_0)-a} - \frac{2\rho_{n_j} g''_{n_j}(\zeta_0)}{g'_{n_j}(\zeta_0)-a\rho_{n_j}}\\ \to& \frac{g''(\zeta_0)}{g(\zeta_0)-a}.\end{aligned}\tag{6}$$

On the other hand,

$$\frac{1}{\rho_n}\frac{(f_n(z_n+\rho_n\zeta_0)-a)(f'_n(z_n+\rho_n\zeta_0)-a)}{f''_n(z_n+\rho_n\zeta_0)} \to \frac{(g(\zeta_0)-a)g'(\zeta_0)}{g''(\zeta_0)}.$$

These two facts imply that

$$\log\frac{|(f_n(z_n+\rho_n\zeta_0)-a)(f'_n(z_n+\rho_n\zeta_0)-a)|}{|f''_n(z_n+\rho_n\zeta_0)\psi(f_n(z_n+\rho_n\zeta_0))^2|}\to -\infty \quad \text{as } n\to\infty. \tag{7}$$

For $n=1,2,\dots$, put

$$h_n(z)=f_n(z_n+\rho_n\zeta_0+z).$$

Let n be sufficiently large. Then h_n is defined and holomorphic on the disk $0<|z|<1/2$, since $z_n+\rho_n\zeta_0\to 0$. By (5) and (6),

$$h_n(0) = g_n(\zeta_0)\to g(\zeta_0)\neq a,0, \tag{8}$$

$$h'_n(0) = \frac{1}{\rho_n}g'_n(\zeta_0)\to\infty, \tag{9}$$

$$h''_n(0) = \frac{1}{\rho_n^2}g''_n(\zeta_0)\to\infty, \tag{10}$$

$$\psi(h_n(0)) = \psi(f_n(z_n+\rho_n\zeta_0))\to\infty. \tag{11}$$

Applying Lemma 3 to $h_n(z)$ and using (7) we get

$$T(r,h_n) \le LD(r,h_n:3,2,3,2) \tag{12}$$

for sufficiently large n. For $1/4 < r < \rho_1 < 1/2$, let $\rho' = (r+\rho)/2$. By Lemma 5, we have

$$\begin{aligned} LD(r,h_n:3,2,3,2) \le M\Big\{1 &+ \log^+\log^+\frac{1}{|h_n(0)-a|} + \log^+\log^+\frac{1}{|h_n'(0)|} \\ &+ \log^+\log^+\frac{1}{|h_n'(0)-a|} + \log^+\frac{1}{\rho'-r} \\ &+ \log^+ T(\rho',h_n) + \log^+ T(\rho',h_n')\Big\}. \end{aligned} \tag{13}$$

Note that

$$\begin{aligned} \log^+ T(\rho',h_n') &\le \log^+ T(\rho',h_n) + \log^+ m\Big(\rho',\frac{h_n'}{h_n}\Big) \\ &\le \log^+ T(\rho,h_n) + m\Big(\rho',\frac{h_n'}{h_n}\Big). \end{aligned} \tag{14}$$

Applying Lemma 5 to $0 < \rho' < \rho$, we have

$$m\Big(\rho',\frac{h_n'}{h_n}\Big) \le M\left\{1 + \log^+\log^+\frac{1}{|h_n(0)|} + \log^+\frac{1}{\rho-\rho'} + \log^+ T(\rho,h_n)\right\}. \tag{15}$$

It follows from (8)–(15) that

$$T(r,h_n) \le b_1 + b_2\log^+\frac{1}{\rho-r} + b_3\log^+ T(\rho,h_n),$$

where b_1, b_2 and b_3 are constants independent of n. By Lemma 6, we obtain

$$T(\frac{1}{4},h_n) \le A,$$

where A is a constant independent of n. Thus $f_n(z)$ are bounded for sufficiently large n and $|z| < 1/8$. However, from

$$\rho_n^2 f_n''(z_n+\rho_n\zeta_0) = g_n''(\zeta_0) \to g''(\zeta_0) \ne 0$$

we see that $f_n(z)$ cannot be bounded in $|z| < 1/8$. Therefore we get a contradiction. The proof is complete. ∎

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